

# On Word Equations Originated from Discrete Dynamical Systems Related to Antisymmetric Cubic Maps With Some Applications

Elias Abboud

Beit Berl College, Doar Beit Berl 44905, Israel  
eabboud@beitberl.ac.il

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## Abstract

In this article, we solve some word equations originated from discrete dynamical systems related to antisymmetric cubic map. These equations emerge when we work with primitive and greatest words. In particular, we characterize all the cases for which  $\langle \beta_1 \overline{\beta_1} \rangle = \langle \beta_2 \overline{\beta_2} \rangle$  where  $\beta_1$  and  $\beta_2$  are the greatest words in  $\langle \langle \beta_1 \rangle \rangle$  and  $\langle \langle \beta_2 \rangle \rangle$  of  $\mathbf{M}(n)$ .

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**Running title:** "Word Equations from Antisymmetric Cubic Maps with Applications"

## 1 Introduction

Words in discrete dynamical systems have been widely studied. Sun and Helmsberg [8] presented an algorithm for recognizing maximality of a word by introducing an extended order of words connected with unimodal maps. Chen and Wang [5] studied the relation between the kneading sequences of unimodal maps and the decomposition of necklaces. Dai et al. [6], [7] gave some combinatorial properties of the periodic kneading sequences of quadratic maps and antisymmetric cubic maps. Lu [10] introduced two extended orders on the kneading sequences of antisymmetric cubic maps and discussed the enumeration of kneading words and the decomposition of corresponding necklaces.

Historically, applied symbolic dynamics was first developed for unimodal maps, which are realized by quadratic maps from the unit interval into itself. One of the major topics which was studied by many authors is the MSS sequences, see [1]-[6]. On the contrary, combinatorics of symbolic dynamics of antisymmetric cubic maps is less investigated (see [7] and [10]).

Word equations are very important when dealing with combinatorics of words. Lothaire [9, p.162] devotes a full chapter on this topic, especially he opens the chapter by presenting the full solution of the equation:  $XY = YX$ . In this case both words  $X$  and  $Y$  must be powers of the same word.

In this article, we solve some word equations originated from discrete dynamical systems related to antisymmetric cubic map. These equations are of the form  $ZW = \overline{WZ}$ ,  $XY = \overline{YX}$  and  $XY = \overline{YZ}$  where  $X, Y, Z, W \in \mathbf{W}$ . In the first two cases we give explicit solutions which implies in particular that some words are not primitive and in the latter case we get that  $XY$  is an alternating broken word. These equations emerge when we work with primitive and greatest words. In particular, we characterize all cases for which  $\langle \beta_1 \overline{\beta_1} \rangle = \langle \beta_2 \overline{\beta_2} \rangle$  where  $\beta_1$  and  $\beta_2$  are the greatest words in  $\langle \langle \beta_1 \rangle \rangle \in \mathbf{M}(n)$  and  $\langle \langle \beta_2 \rangle \rangle \in \mathbf{M}(n)$ , respectively, (see Theorems 1 and 2 below).

## 2 Preliminaries

We borrow the main notation and terminology from [10]. Corresponding to the symbolic dynamics of antisymmetric cubic maps, we shall be concerned with the admissible sequences which are (1) all infinite sequences on  $\{L, M, R\}$  (2) all finite sequences of the forms  $\gamma \overline{C}$  and  $\gamma C \overline{\gamma C}$ , where  $\gamma$  is a word on  $\{L, M, R\}$ . Admissible sequences are ordered by the parity-lexicographic order. The letters  $\{L, \overline{C}, M, C, R\}$  are endowed with the order  $L < \overline{C} < M < C < R$ . A finite sequence is said to be *even* (*odd*) if it contains an even (odd) number of  $M$ 's. Let  $w = w_1 \dots w_k s \dots$  and  $v = w_1 \dots w_k t \dots$  be two admissible sequences such that  $s \neq t$ . The order relation of  $w$  and  $v$  is defined as follows: when  $w_1 \dots w_k$  is even then  $w > v$  if  $s > t$  and  $w < v$  if  $s < t$ ; otherwise, when  $w_1 \dots w_k$  is odd then  $w < v$  if  $s > t$  and  $w > v$  if  $s < t$ .

Following [9] for the terminology and notations for words, let  $A$  be a finite alphabet, whose elements are called letters. A word  $\alpha$  on  $A$  is a finite sequence of elements of  $A$ :  $a_1 a_2 \dots a_n$  where  $n$  is called the *length* of  $\alpha$ , denoted by  $|\alpha|$ . The set of all words on  $A$  is denoted by  $A^*$ . The set  $A^*$  is equipped with a binary operation obtained by  $(a_1 a_2 \dots a_n)(b_1 b_2 \dots b_m) = a_1 a_2 \dots a_n b_1 b_2 \dots b_m$ . The empty word is the identity of this operation. Given a word  $w \in A^*$  and  $n \geq 1$ , by  $w^n$  we denote the word  $ww \dots w$  ( $n$  terms). A word  $v \in A^*$  is said to be a *left* or a *right factor* of a word  $x \in A^*$  if  $x = vx_1$  or  $x = x_2 v$ , respectively, where  $x_1, x_2 \in A^*$ . Denote by  $v|x$  when  $v$  is a left factor of  $x$  and by  $v \nmid x$  when  $v$  is not a left factor of  $x$ .

Define

$$\mathbf{W}_n = \{t_1 t_2 \dots t_n | t_i = L, M \text{ or } R\},$$

$$\mathbf{W} = \bigcup_{n \geq 0} \mathbf{W}_n.$$

For a word  $w = w_1 \dots w_k \in \mathbf{W}$ , we define the *complementation*  $\overline{w}$  as  $\overline{w} = \overline{w_1 \dots w_k}$ , where  $\overline{L} = R$ ,  $\overline{L} = R$  and  $\overline{M} = M$ . A word  $w \in \mathbf{W}$  of length  $n$  is said to be *primitive* provided its smallest subperiod is also of length  $n$ , i.e.,  $w \neq u^l, l \geq 2$ .

For  $w \in \mathbf{W}$ , we denote by  $\langle w \rangle$  the set of words which can be obtained from  $w$  by cyclic permutations, and call  $\langle w \rangle$  a *conjugate class* or a *necklace*. Let  $\langle\langle w \rangle\rangle = \langle w \rangle \cup \langle \overline{w} \rangle$ . We call a necklace  $\langle w \rangle$  *self-complementary* if  $\overline{w} \in \langle w \rangle$ . It is clear that  $\langle w \rangle$  is self-complementary if and only if  $\langle\langle w \rangle\rangle = \langle w \rangle$ .

Define also the special sets of words:

$$\mathbf{M}(n) = \{ \langle\langle w \rangle\rangle = \langle w \rangle \cup \langle \overline{w} \rangle \mid w \in \mathbf{W}_n \text{ is primitive} \},$$

$$\mathbf{U}(n) = \{ \langle w \rangle \mid w \in \mathbf{W}_n \text{ is primitive and self-complementary} \}.$$

Lu [10], introduced an extended parity-lexicographic order on the words in  $\mathbf{W}$ . Since there are two critical points  $C$  and  $\overline{C}$ , he introduced two such orders, defined as follows: Let  $u$  and  $v$  be words in  $\mathbf{W}$ . If  $u \nmid w$  and  $w \nmid u$  then  $u$  and  $w$  are ordered according to the parity-lexicographic order. However, if  $u|w$ , say  $w = uv$  then

- a. The  $\overline{C}$ -order:  $uv > u$  if  $u$  is odd, otherwise  $u > uv$ .
- b. The  $C$ -order:  $uv > u$  if  $u$  is even, otherwise  $u > uv$ .

If  $D$  denotes  $\overline{C}$  or  $C$  then given  $w \in \mathbf{W}$ , we call  $w$  a  $D$ -lexical word if  $w$  is greater than all of its right shifts (right factors) and  $\overline{w}$  is less than all of its right shifts in  $D$ -order.

Let  $L_1(D) = \{M, R\}$ , and for  $n > 1$ , let

$$\mathbf{L}_n(D) = \{w \mid w \in \mathbf{W}_n \text{ and } w \text{ is } D\text{-lexical}\}.$$

**Definition 1** A word  $w \in \mathbf{W}$  is called a broken alternating word if it is of the form  $w = (w_1 \overline{w_1})^n w_0$  where  $n$  is a positive integer and  $w_0$  is a left factor of  $w_1 \overline{w_1}$ .

We need the following elementary properties (see Lemmas 2.1.2.2 and 2.3 in [Lu]);

**Lemma 1** (a) Let  $w = w_1 w_2 \dots w_n \in \mathbf{L}_n(C)$ ,  $n > 1$ . Then  $w_1 = R$  and  $w_n = M$  or  $R$ .

(b) Let  $\alpha, \beta, \gamma$  and  $w$  be words in  $\mathbf{W}$ . If  $\alpha\beta < w < \alpha\gamma$  in  $\overline{C}$ -order or  $C$ -order, then  $\alpha|w$ .

(c) Suppose that  $\alpha$  and  $\beta$  are words in  $\mathbf{W}$  such that  $\alpha \nmid \beta$  and  $\beta \nmid \alpha$ . Then in  $\overline{C}$ -order or  $C$ -order we have;

$$\alpha > \beta \Leftrightarrow \overline{\beta} > \overline{\alpha}$$

### 3 Word equations on $\mathbf{W}$

The aim of this section is to solve the word equations  $ZW = \overline{WZ}$ ,  $XY = \overline{YX}$  and  $XY = \overline{YZ}$  where  $X, Y, Z, W \in \mathbf{W}$ . We shall assume that all words are not empty.

### 3.1 The word equation $ZW = \overline{WZ}$

**Proposition 1** Suppose  $ZW = \overline{WZ}$  where  $|ZW| = m$  and  $|W| = r$ . Let  $d = (m, r)$  be the greatest common divisor. Then one of the following occurs:

- (1)  $|Z| = |W|$  implies  $Z = \overline{W}$ .
- (2)  $|Z| \neq |W|$  implies (a)  $Z = (\overline{L}L)^{(\frac{m-r}{d}-1)/2}\overline{L}$  and  $W = (L\overline{L})^{(\frac{r}{d}-1)/2}L$ , if  $\frac{m}{d}$  is even, where  $\overline{L}$  is a left factor of  $Z$  of length  $d$ . Hence,  $Z$  and  $W$  are alternating broken words.  
Or (b)  $ZW = M^m$  if  $\frac{m}{d}$  is odd.

**Proof.** (1) Obvious.

(2) Suppose  $|ZW| = m$  and  $|W| = r$ , then  $r < m$ . Let  $Z = \overline{y_1 y_2 \dots y_{m-r}}$  and  $W = \overline{y_{m-r+1} \dots y_m}$ .

It follows that

$$ZW = \overline{y_1 y_2 \dots y_m} = y_{m-r+1} \dots y_m y_1 y_2 \dots y_{m-r}. \quad (1)$$

As a result, we get the relation;

$$\overline{y_i} = y_j, 1 \leq i \leq m,$$

where

$$m - r + i \equiv j \pmod{m}, 1 \leq j \leq m.$$

Denote  $t = m - r$  then obviously  $d = (m, r)|t$ .

Subcase 1:  $\frac{m}{d}$  is odd.

Clearly we have;

$$\overline{y_i} = y_{i+t} = \overline{y_{i+2t}} = \dots = y_{i+(2k+1)t},$$

where all indices are taken mod  $m$ . Now we ask whether there exists a  $k$  for which

$$i + (2k+1)t \equiv i \pmod{m}.$$

This congruence is equivalent to the following:

$$(2k+1)t \equiv 0 \pmod{m}.$$

Thus it is sufficient to take  $2k+1 = \frac{m}{d}$ , so that

$$(2k+1)t \equiv \frac{m}{d}t \equiv m \frac{t}{d} \equiv 0 \pmod{m}.$$

This computation can be carried out for each  $i$  and therefore  $\overline{y_i} = y_i$  for every  $i, 1 \leq i \leq m$ . Hence,  $y_i = M$ , and  $ZW = M^m$ .

Subcase 2:  $\frac{m}{d}$  is even.

In this case the following is satisfied for every  $1 \leq i \leq m$ ;

$$\overline{y_i} = y_{i+t} = \overline{y_{i+2t}} = \dots = \overline{y_{i+\frac{m}{d}t}}. \quad (2)$$

Clearly, the group  $\mathbb{Z}_m$  acts on the set  $S = \{\overline{y_1}, \overline{y_2}, \dots, \overline{y_m}, y_1, y_2, \dots, y_m\}$  in the following manner: for  $a \in \mathbb{Z}_m$  define  $\overline{y_i}^a = y_{i+at}$  and  $y_i^a = \overline{y_{i+at}}$ . Then  $\overline{y_i}^a = \overline{y_i}^b \Leftrightarrow y_{t+i+at} = y_{t+i+bt} \Leftrightarrow t+i+at \equiv t+i+bt \pmod{m}$ . Hence,  $at \equiv bt \pmod{m}$  and since  $t = m - r$  we have  $ar \equiv br \pmod{m}$  which is equivalent to  $a \frac{r}{d} \equiv b \frac{r}{d} \pmod{\frac{m}{d}}$ . But  $(\frac{r}{d}, \frac{m}{d}) = 1$  so that  $a \equiv b \pmod{\frac{m}{d}}$ . This proves that the length of each orbit is exactly  $\frac{m}{d}$  and each orbit which begins with  $\overline{y_i}$  is given in 2, namely;  $(\overline{y_i}, y_{i+t}, \overline{y_{i+2t}}, \dots, y_{i+(\frac{m}{d}-1)t})$ . On the other hand, if  $b = 0$  then  $\overline{y_i}^a = \overline{y_i}^b = y_{t+i} = \overline{y_i}$  and this true iff  $a \equiv 0 \pmod{\frac{m}{d}}$ . Consequently, fixed points occur only under the action of the subset  $\{0, \frac{m}{d}, 2\frac{m}{d}, \dots, (d-1)\frac{m}{d}\} \subseteq \mathbb{Z}_m$  and each has exactly  $m$  fixed points. By the Burnside lemma the number of orbits, which begins with  $\overline{y_i}$ , of the action of  $\mathbb{Z}_m$  on  $S$  is:

$$\begin{aligned} \# \text{orbits} &= \frac{1}{|\mathbb{Z}_m|} \sum_{a \in \mathbb{Z}_m} |Fix(a)| \\ &= \frac{1}{m} [|Fix(0)| + |Fix(\frac{m}{d})| + \dots + |Fix((d-1)\frac{m}{d})|] \\ &= \frac{1}{m} dm = d. \end{aligned}$$

As a result we can arrange the distinct orbits which begin with  $\overline{y_i}$  as follows:

$$\begin{array}{ccccccc} \overline{y_1} & = & y_{1+t} & = & \overline{y_{1+2t}} & = & \dots = y_{1+(\frac{m}{d}-1)t} \\ \overline{y_2} & = & y_{2+t} & = & \overline{y_{2+2t}} & = & \dots = y_{2+(\frac{m}{d}-1)t} \\ & \vdots & & & & & \\ \overline{y_d} & = & y_{d+t} & = & \overline{y_{d+2t}} & = & \dots = y_{d+(\frac{m}{d}-1)t} \end{array}$$

Letting  $L = y_1 y_2 \dots y_d$  then  $\overline{ZW} = y_1 y_2 \dots y_m$  can be partitioned into  $\frac{m}{d}$  (even) alternating orbits of the form  $(L\overline{L})^{\frac{m}{2d}}$  and  $Z = \overline{y_1 y_2 \dots y_{m-r}}$  can be partitioned into  $\frac{m-r}{d}$  (note that  $(\frac{r}{d}, \frac{m}{d}) = 1$  and  $\frac{m}{d}$  is even hence  $\frac{r}{d}$  and  $\frac{m-r}{d}$  are odds) alternating orbits of the form  $(L\overline{L})^{(\frac{m-r}{d}-1)/2} L$ . Therefore;

$$\overline{ZW} = y_1 y_2 \dots y_m = (L\overline{L})^{\frac{m}{2d}} = (L\overline{L})^{n_1}$$

and

$$Z = \overline{y_1 y_2 \dots y_{m-r}} = (\overline{L}L)^{(\frac{m-r}{d}-1)/2} \overline{L} = (\overline{L}L)^{n_2} \overline{L}$$

where  $n_2 = (\frac{m-r}{d} - 1)/2$ .

Hence,

$$W = (L\overline{L})^{(\frac{r}{d}-1)/2} L,$$

and the result follows. ■

**Example 1** ( $m = 6, r = 4$ )

In this case  $d = (m, r) = 2$  and  $\frac{m}{d} = 3$  is odd. Equation (1) becomes

$$\overline{y_1 y_2 y_3 y_4 y_5 y_6} = y_3 y_4 y_5 y_6 y_1 y_2.$$

Therefore,  $\overline{y_1} = y_3 = \overline{y_5} = y_1$ . Likewise,  $\overline{y_2} = y_4 = \overline{y_6} = y_2$ . Continuing this manner one can easily see that  $y_i = M$  for every  $1 \leq i \leq 6$ , which is consistent with the conclusion of the previous proposition.

**Example 2** ( $m = 8, r = 2$ )

In this case  $d = (m, r) = 2$  and  $\frac{m}{d} = 4$  is even. Equation (1) is equivalent to:

$$\overline{y_1 y_2 y_3 y_4 y_5 y_6 y_7 y_8} = y_7 y_8 y_1 y_2 y_3 y_4 y_5 y_6.$$

Hence,

$$\begin{aligned}\overline{y_1} &= y_7 = \overline{y_5} = y_3 = \overline{y_1}, \\ \overline{y_2} &= y_8 = \overline{y_6} = y_4 = \overline{y_2}.\end{aligned}$$

Letting  $L = y_1 y_2$  then  $\overline{ZW} = (y_1 y_2)(y_3 y_4)(y_5 y_6)(y_7 y_8) = (L\overline{L})^2$ , and

$$\begin{aligned}Z &= \overline{y_1 y_2 y_3 y_4 y_5 y_6} = \overline{L L \overline{L}} \\ W &= \overline{y_7 y_8} = L,\end{aligned}$$

which is consistent with the conclusion of the previous proposition since  $\frac{m}{2d} = 2$  and  $\frac{r}{d} = 1$ .

### 3.2 The word equation $XY = \overline{Y}X$

**Proposition 2** Suppose  $XY = \overline{Y}X$  then the words  $XY\overline{XY}, \overline{Y}XY\overline{X}$  are not primitive.

**Proof.** Consider the following cases:

(1) If  $|X| = |Y| = l$  then  $X = \overline{Y}$  and  $Y = X$  so that  $X = \overline{X}$  and  $Y = \overline{Y}$ . Equivalently,  $X = Y = M^l$ . Hence, all words on  $X$  and  $Y$  which contain at least two letters are non-primitive.

(2)  $|X| = l > |Y| = m$ .

Let  $l = qm + r$ ,  $0 \leq r < m$ , and denote  $X = x_1 x_2 \dots x_l$  and  $Y = y_1 y_2 \dots y_m$  where  $x_i, y_j \in \{L, M, R\}$ ,  $1 \leq i \leq l$ ,  $1 \leq j \leq m$ . Then the word equation  $XY = \overline{Y}X$  is equivalent to:

$$x_1 x_2 \dots x_l y_1 y_2 \dots y_m = \overline{y_1 y_2 \dots y_m} x_1 x_2 \dots x_l.$$

Consequently, comparing the letters of  $X$  in both sides and taking in account the periodicity we have;

$$X = (\overline{y_1 y_2 \dots y_m})^q \overline{y_1 y_2 \dots y_r} = y_{m-r+1} \dots y_m (y_1 y_2 \dots y_m)^q.$$

Since  $l > m$  then  $q \geq 1$ . It follows that

$$\overline{y_1 y_2 \dots y_m} = y_{m-r+1} \dots y_m y_1 y_2 \dots y_{m-r}.$$

Let  $Z = \overline{y_1 y_2 \dots y_{m-r}}$  and  $W = \overline{y_{m-r+1} \dots y_m}$  then the equation which was discussed in the previous proposition is satisfied:  $ZW = \overline{WZ}$ , where  $|ZW| = m$  and  $|W| = r$ . Let  $d = (m, r)$  then we have two cases:

If  $\frac{m}{d}$  is odd then  $Y = \overline{ZW} = M^m$  and  $X = M^l$ . In particular, the word  $XY\overline{XY}$  is not primitive.

If  $\frac{m}{d}$  is even then  $Y = \overline{ZW} = (\overline{LL})^{\frac{m}{2d}} = (\overline{LL})^{n_1}, n_1 \geq 1$ .  
and

$$X = (\overline{y_1 y_2 \dots y_m})^q \overline{y_1 y_2 \dots y_r} = (\overline{LL})^{\frac{m}{2d}q} (\overline{LL})^{(\frac{r}{d}-1)/2} \overline{L} = (\overline{LL})^{n_2} \overline{L}$$

where  $n_2 = \frac{m}{2d}q + (\frac{r}{d} - 1)/2 \geq 1$ .

In particular, the words

$$\begin{aligned} XY\overline{XY} &= (\overline{LL})^{2n_1+2n_2+1}, \\ \overline{Y}XY\overline{X} &= (\overline{LL})^{2n_1+2n_2+1}, \end{aligned}$$

are not primitive.

(3)  $|X| = l < |Y| = m$ .

The equation  $XY = \overline{Y}X$  implies that  $X$  is a left factor of  $\overline{Y}$  and  $X$  is a right factor of  $Y$ . Hence,  $\overline{Y} = XY_1$  and  $Y = Y_2X$ . Substituting back in the equation we get  $Y_1 = Y_2$  and therefore  $\overline{Y} = XY_1$  and  $Y = Y_1X$ . Equivalently,  $\overline{XY_1} = Y_1X$ . Apply Proposition 1 by taking  $Z = Y_1, W = X$  then  $Y = Y_1X = ZW, |Y| = |ZW| = m$  and  $|X| = |W| = l$ . Let  $d = (m, l)$ . If  $\frac{m}{d}$  is odd then  $Y = ZW = M^m$  and  $X = W = M^l$  and therefore  $XY\overline{XY}$  and  $\overline{Y}XY\overline{X}$  are not primitive. If  $\frac{m}{d}$  is even then  $Z = (\overline{LL})^{(\frac{m-l}{d}-1)/2} \overline{L}$  and  $X = W = (\overline{LL})^{(\frac{l}{d}-1)/2} \overline{L}$  so that  $Y = ZW = (\overline{LL})^{n_1}, n_1 \geq 1, XY = (\overline{LL})^{n_2} \overline{L}, n_2 \geq 1$ , and  $XY\overline{XY} = (\overline{LL})^{n_2} \overline{L} (\overline{LL})^{n_2} \overline{L} = (\overline{LL})^{2n_2+1}$  is not primitive.

The word  $\overline{Y}XY\overline{X}$  is also not primitive. Since,  $\overline{Y}X = (\overline{LL})^{n_1} (\overline{LL})^{(\frac{l}{d}-1)/2} \overline{L} = (\overline{LL})^{n_3} \overline{L}, n_3 \geq 1$  and  $\overline{Y}XY\overline{X} = (\overline{LL})^{n_3} \overline{L} (\overline{LL})^{n_3} \overline{L} = (\overline{LL})^{2n_3+1}$ . ■

### 3.3 The word equation $XY = \overline{Y}Z$

**Proposition 3** Suppose  $XY = \overline{Y}Z$  then;

- (1)  $|X| = |Y| = |Z|$  implies  $X = \overline{Z}$  and the equation becomes  $XY = \overline{Y}X$ .
- (2) If  $|X| = |Z| < |Y|$  then denote  $|Y| = q|X| + r$ . We have two cases: (a) for even  $q$ , it follows that  $\overline{X} = Y_0X_0, Z = \overline{X_0}Y_0$  and  $Y = (Y_0X_0\overline{Y_0X_0})^{\frac{q}{2}}Y_0$  is a broken alternating word. (b) for odd  $q$ , it follows that  $X = Y_0X_0, Z = \overline{X_0}Y_0$  and  $Y = (\overline{Y_0X_0}Y_0X_0)^{\frac{q-1}{2}}\overline{Y_0X_0}Y_0$  is a broken alternating word.
- (3)  $|X| = |Z| > |Y|$  implies  $X = \overline{Y}X_1$  and  $Z = X_1Y$ .

**Proof.** There are three cases, namely;

(1)  $|X| = |Y| = |Z|$ . In this case  $X = \overline{Y}$  and  $Y = Z$  hence,  $X = \overline{Z}$  and the equation becomes  $XY = \overline{Y}X$ . The conclusion of Proposition 1 applies.

(2)  $|X| = |Z| < |Y|$ . Suppose  $|Y| = q|X| + r$  and partition  $Y$  into subwords as follows:  $Y = Y_1Y_2\dots Y_qY_0$  where,  $|Y_i| = |X|, 1 \leq i \leq q$  and  $|Y_0| = r$ . Hence, the equation  $XY = \overline{Y}Z$  is equivalent to the equation:

$$XY_1Y_2\dots Y_qY_0 = \overline{Y_1Y_2\dots Y_qY_0}Z.$$

Consequently,  $X = \overline{Y_1}, Y_1 = \overline{Y_2}, \dots, Y_{q-1} = \overline{Y_q}$  and  $Y_qY_0 = \overline{Y_0}Z$ . Moreover,  $Y_q = X$  or  $Y_q = \overline{X}$  according to whether  $q$  is even or odd, respectively. Thus

we have:

$$Y = \begin{cases} (\overline{X}X)^{\frac{q}{2}}Y_0, & q \text{ is even and } Y_0 \text{ is a left factor of } \overline{Y}_q = \overline{X} \\ (\overline{X}X)^{\frac{q-1}{2}}\overline{X}Y_0, & q \text{ is odd and } Y_0 \text{ is a left factor of } \overline{Y}_q = X \end{cases}.$$

Consider two subcases;

Subcase 1: If  $q$  is even, write  $\overline{X} = Y_0X_0$  then  $XY_0 = \overline{Y_0X_0}Y_0 = \overline{Y_0}Z$  implies  $Z = \overline{X_0}Y_0$  and  $Y = (Y_0X_0\overline{Y_0X_0})^{\frac{q}{2}}Y_0$

Subcase 2: If  $q$  is odd, write  $X = Y_0X_0$  then  $\overline{X}Y_0 = \overline{Y_0X_0}Y_0 = \overline{Y_0}Z$  implies  $Z = \overline{X_0}Y_0$  and  $Y = (\overline{Y_0X_0}Y_0X_0)^{\frac{q-1}{2}}\overline{Y_0X_0}Y_0$ .

(3)  $|X| = |Z| > |Y|$ . The best possible we can get from the equation  $XY = \overline{Y}Z$  is the following:  $X = \overline{Y}X_1$  and  $Z = Z_1Y$ . It is easily seen that  $X_1 = Z_1$  and therefore  $X = \overline{Y}X_1$  and  $Z = X_1Y$ . ■

## 4 Applications

Firstly, we prove that  $\beta\overline{\beta}$  is a primitive word provided  $\beta$  is primitive and is the greatest word in  $\langle\langle\beta\rangle\rangle$ .

**Lemma 2** *If  $\beta \in \mathbf{L}_n(C)$  or  $\beta \in \mathbf{L}_n(\overline{C})$  is primitive and is the greatest word in  $\langle\langle\beta\rangle\rangle$ , then  $\beta\overline{\beta}$  is primitive.*

**Proof.** Suppose that  $\beta\overline{\beta}$  is not primitive. Then  $\beta\overline{\beta} = (\gamma\overline{\gamma})^k, k > 1$ . We have two cases;

Case 1.  $k = 2m, m \geq 1$ .

In this case,  $\beta\overline{\beta} = (\gamma\overline{\gamma})^{2m}$  implies that  $\beta = (\gamma\overline{\gamma})^m = \overline{\beta}$ . This implies that  $\beta = M^l, l = 2|\gamma|m \geq 2$ , and since  $\beta$  is primitive we get a contradiction.

Case 2.  $k = 2m + 1, m \geq 1$ .

In this case,  $\beta\overline{\beta} = (\gamma\overline{\gamma})^{2m+1}$  implies that  $\beta = (\gamma\overline{\gamma})^m\gamma, m \geq 1$ . Consider the following subcases:

Subcase I.  $\gamma = L\gamma_1$ . Let  $\beta' = \overline{\gamma}\gamma(\gamma\overline{\gamma})^{m-1}\gamma \in \langle\beta\rangle$  then  $\beta' > \beta$ . This is true since  $\beta$  begins with  $L$  and  $\beta'$  begins with  $R$ . Thus we have a contradiction.

Subcase II.  $\gamma = R\gamma_1$  and  $\gamma$  is even. Let  $\beta'' = \gamma(\gamma\overline{\gamma})^m \in \langle\beta\rangle$  then  $\beta'' > \beta$ . This is true since  $\gamma L$  is a left factor of  $\beta$ , while  $\gamma R$  is a left factor of  $\beta''$ . Thus we have a contradiction.

Subcase III.  $\gamma = R\gamma_1$  and  $\gamma$  is odd.

Now, suppose that  $\beta \in \mathbf{L}_n(C)$ . By the definition of the  $C$ -order and since  $\gamma$  is odd we have:

$$\beta = (\gamma\overline{\gamma})^m\gamma < \gamma.$$

But this contradicts that  $\beta$  is  $C$ -lexical, which asserts that  $\beta$  is greater than all of its right shifts. If  $\beta \in \mathbf{L}_n(\overline{C})$ , then

$$\overline{\beta} = (\overline{\gamma}\gamma)^m\overline{\gamma}.$$

Since  $\overline{\gamma}$  is odd, then in the  $\overline{C}$ -order we have  $\overline{\beta} > \overline{\gamma}$ . But this contradicts the lexicality of  $\beta \in \mathbf{L}_n(\overline{C})$ , which asserts that  $\beta$  is greater than all of its right shifts and  $\overline{\beta}$  is less than all of its right shifts.

The result follows. ■



**Lemma 3** Let  $\beta \in \mathbf{W}_n$ . If  $\beta$  is primitive and is the greatest word in  $\langle\langle\beta\rangle\rangle$  then  $\beta\bar{\beta}$  is primitive.

**Proof.** If  $\beta \in \mathbf{L}_n(C)$  or  $\beta \in \mathbf{L}_n(\bar{C})$  then by the previous lemma the result follows. Suppose  $\beta \notin \mathbf{L}_n(C)$  and  $\beta \notin \mathbf{L}_n(\bar{C})$ . Hence, by Theorem 2.8 [Lu],  $\beta = \delta\bar{\delta}$  for odd  $\delta$ . On the other hand, by the discussion in the previous lemma there is a positive integer  $m$  such that  $\beta\bar{\beta} = (\gamma\bar{\gamma})^{2m+1}$ , for which  $\gamma$  is odd and this implies that

$$\beta = (\gamma\bar{\gamma})^m\gamma, m \geq 1.$$

Consider the following cases:

Case I.  $m = 2t, t \geq 1$ .

In this case,

$$\beta = (\gamma\bar{\gamma})^t\gamma(\bar{\gamma}\gamma)^t = \delta\bar{\delta}.$$

This equation yields that  $|\gamma|$  is even. Thus we may write  $\gamma = \lambda_1\lambda_2$ ,  $|\lambda_1| = |\lambda_2|$ . Hence,  $\delta = (\gamma\bar{\gamma})^t\lambda_1$  and  $\bar{\delta} = \lambda_2(\bar{\gamma}\gamma)^t$ . Therefore,  $\lambda_1 = \bar{\lambda}_2$  and hence  $\gamma = \lambda_1\bar{\lambda}_1$ . But this contradicts the fact that  $\gamma$  is an odd word.

Case II.  $m = 2t + 1, t \geq 0$ .

In this case,

$$\beta = (\gamma\bar{\gamma})^t\gamma\bar{\gamma}\gamma(\bar{\gamma}\gamma)^t = \delta\bar{\delta}.$$

Letting  $\gamma = \lambda_1\lambda_2$ ,  $|\lambda_1| = |\lambda_2|$ , we get  $\delta = (\gamma\bar{\gamma})^t\lambda_1\lambda_2\bar{\lambda}_1$  and  $\bar{\delta} = \bar{\lambda}_2\lambda_1\lambda_2(\bar{\gamma}\gamma)^t$  which implies that  $\lambda_1 = \bar{\lambda}_2$  and  $\lambda_1 = \lambda_2$ . But then  $\lambda_1 = \lambda_2 = M$  and  $\beta$  is not primitive, a contradiction. ■

The next theorem characterizes all cases for which  $\langle\beta_1\bar{\beta}_1\rangle = \langle\beta_2\bar{\beta}_2\rangle$  where  $\beta_1$  and  $\beta_2$  are the greatest words in  $\langle\langle\beta_1\rangle\rangle \in \mathbf{M}(n)$  and  $\langle\langle\beta_2\rangle\rangle \in \mathbf{M}(n)$ .

**Theorem 1** If  $\langle\beta_1\bar{\beta}_1\rangle = \langle\beta_2\bar{\beta}_2\rangle$  where  $\beta_1$  and  $\beta_2$  are the greatest words in  $\langle\langle\beta_1\rangle\rangle \in \mathbf{M}(n)$  and  $\langle\langle\beta_2\rangle\rangle \in \mathbf{M}(n)$ , respectively, then one of the following occurs:

- (1)  $\langle\langle\beta_1\rangle\rangle = \langle\langle\beta_2\rangle\rangle$ .
- (2)  $\beta_2 = \lambda\mu \in \mathbf{L}_n(\bar{C}), \beta_1 = \bar{\mu}\lambda \in \mathbf{L}_n(\bar{C})$  and  $\lambda\mu = \bar{\mu}\eta$ .
- (3)  $\beta_1 = \bar{\mu}\lambda_1\mu\bar{\lambda}_1, \beta_2 = \lambda_1\mu\bar{\lambda}_1\mu \in \mathbf{L}_n(\bar{C})$  and  $\lambda_1\mu = \bar{\mu}\lambda_1$ .
- (4)  $\beta_2 = \lambda\mu_1\bar{\lambda}\bar{\mu}_1 \in \mathbf{L}_n(\bar{C}), \beta_1 = \bar{\mu}_1\bar{\lambda}\mu_1\lambda$  and  $\lambda\mu_1 = \bar{\mu}_1\bar{\lambda}$ .
- (5)  $\beta_1 = \bar{\mu}_1\bar{\lambda}\mu_1\lambda, \beta_2 = \lambda\mu_1\bar{\lambda}\bar{\mu}_1 \in \mathbf{L}_n(\bar{C})$  and  $\bar{\mu}_1\bar{\lambda} = \lambda\eta$ .
- (6)  $\beta_1 = \bar{\mu}\lambda_1\mu\lambda_1 \in \mathbf{L}_n(\bar{C}), \beta_2 = \bar{\lambda}_1\bar{\mu}\lambda_1\mu$  and  $\bar{\mu}\lambda_1 = \bar{\lambda}_1\bar{\mu}$ .
- (7)  $\beta_1 = \bar{\mu}_1\lambda\mu_1\lambda \in \mathbf{L}_n(\bar{C}), \beta_2 = \lambda\mu_1\bar{\lambda}\bar{\mu}_1$  and  $\lambda\mu_1 = \bar{\mu}_1\lambda$ .

**Proof.** Suppose

$$\langle\beta_1\bar{\beta}_1\rangle = \langle\beta_2\bar{\beta}_2\rangle,$$

then  $\beta_1\bar{\beta}_1 \in \langle\beta_2\bar{\beta}_2\rangle$ . Denote  $\beta_1 = u_1u_2\dots u_n$  and  $\beta_2 = v_1v_2\dots v_n$ . Now, If  $\beta_1 \neq \beta_2$  and  $\beta_1 \neq \bar{\beta}_2$  then;

$$\beta_1\bar{\beta}_1 = u_1u_2\dots u_n\bar{u}_1\bar{u}_2\dots\bar{u}_n = \bar{v}_{k+1}\dots\bar{v}_nv_1v_2\dots v_n\bar{v}_1\dots\bar{v}_k,$$

or

$$\beta_1\bar{\beta}_1 = u_1u_2\dots u_n\bar{u}_1\bar{u}_2\dots\bar{u}_n = v_{k+1}v_{k+2}\dots v_n\bar{v}_1\bar{v}_2\dots\bar{v}_nv_1v_2\dots v_k,$$

where  $0 < k < n$ .

Let  $\lambda = v_1 v_2 \dots v_k$  and  $\mu = v_{k+1} v_{k+2} \dots v_n$  then  $\beta_1 = \bar{\mu} \lambda$  and  $\beta_2 = \lambda \mu$  or  $\beta_1 = \mu \bar{\lambda}$  and  $\beta_2 = \lambda \mu$ .

Since both cases are similar we shall consider only the first case, namely;  $\lambda = v_1 v_2 \dots v_k$  and  $\mu = v_{k+1} v_{k+2} \dots v_n$ ,  $0 < k < n$ , and  $\beta_1 = \bar{\mu} \lambda$  and  $\beta_2 = \lambda \mu$ . If  $\beta_1 | \beta_2$  or  $\beta_2 | \beta_1$  then  $\beta_1 = \beta_2$ , because both words have the same length. Hence, case (1) of the theorem is satisfied. Assume therefore that  $\beta_1 \nmid \beta_2$  and  $\beta_2 \nmid \beta_1$ .

We have the following subcases;

I.  $\beta_1 \in \mathbf{L}_n(\bar{C})$  and  $\beta_2 \in \mathbf{L}_n(\bar{C})$ .

Suppose first that  $\beta_1 < \beta_2$  in  $\bar{C}$ -order. If  $\lambda$  is even then

$$\beta_1 = \bar{\mu} \lambda < \beta_2 = \lambda \mu < \lambda.$$

But this contradicts the lexicality of  $\beta_1 \in \mathbf{L}_n(\bar{C})$ , which asserts that  $\beta_1$  is greater than all of its right shifts. If  $\lambda$  is odd then, by Lemma 1 (c),

$$\bar{\beta}_1 = \mu \bar{\lambda} > \bar{\beta}_2 = \bar{\lambda} \bar{\mu} > \bar{\lambda}.$$

This also contradicts the lexicality of  $\beta_1 \in \mathbf{L}_n(\bar{C})$ , which asserts that  $\beta_1$  is greater than all of its right shifts and  $\bar{\beta}$  is less than all of its right shifts.

On the other hand, if  $\beta_2 < \beta_1$  in  $\bar{C}$ -order then;

$$\beta_2 = \lambda \mu < \beta_1 = \bar{\mu} \lambda.$$

Since  $\beta_2 = \lambda \mu$  is greatest in  $\langle\langle \beta_2 \rangle\rangle$  and  $\bar{\mu} \bar{\lambda} \in \langle\langle \beta_2 \rangle\rangle$  then we have:

$$\bar{\mu} \bar{\lambda} < \beta_2 = \lambda \mu < \beta_1 = \bar{\mu} \lambda.$$

Hence,  $\bar{\mu} | \lambda \mu$  and we get the word equation;

$$\lambda \mu = \bar{\mu} \eta \tag{3}$$

Thus, case (2) of the theorem is satisfied.

II.  $\beta_1 \notin \mathbf{L}_n(\bar{C})$  or  $\beta_2 \notin \mathbf{L}_n(\bar{C})$  but not both.

Since  $\beta_1$  and  $\beta_2$  are primitive and the greatest words in  $\langle\langle \beta_1 \rangle\rangle$  and  $\langle\langle \beta_2 \rangle\rangle$ , respectively, then by Theorem 2.8 [Lu],  $\beta_1 = \delta \bar{\delta}$  or  $\beta_2 = \delta \bar{\delta}$  for odd  $\delta$ .

Subcase 1:  $\beta_1 = \bar{\mu} \lambda = \delta \bar{\delta}$  for odd  $\delta$  and  $\beta_2 = \lambda \mu \in \mathbf{L}_n(\bar{C})$ .

(a) If  $|\mu| = |\delta|$  then  $\bar{\mu} = \delta$  and  $\lambda = \bar{\delta}$  and hence,  $\lambda = \mu$ . But then  $\beta_2 = \lambda^2$  which is a contradiction to the fact that  $\beta_2$  is primitive.

(b) If  $|\mu| < |\delta|$  then  $|\lambda| > |\delta|$  and therefore  $\delta = \bar{\mu} \delta_1$  and  $\lambda = \lambda_1 \bar{\delta}$ . Substituting in the relation  $\bar{\mu} \lambda = \delta \bar{\delta}$  we get  $\delta_1 = \lambda_1$ . Consequently;  $\beta_1 = \bar{\mu} \lambda_1 \mu \bar{\lambda}_1$  and  $\beta_2 = \lambda_1 \mu \bar{\lambda}_1 \mu$ . Suppose firstly that  $\beta_1 = \bar{\mu} \lambda_1 \mu \bar{\lambda}_1 < \beta_2 = \lambda_1 \mu \bar{\lambda}_1 \mu$ . Since  $\beta_1$  is the greatest word in  $\langle\langle \beta_1 \rangle\rangle$  and  $\lambda_1 \mu \bar{\lambda}_1 \bar{\mu} \in \langle\langle \beta_1 \rangle\rangle$  we have:

$$\lambda_1 \mu \bar{\lambda}_1 \bar{\mu} < \beta_1 = \bar{\mu} \lambda_1 \mu \bar{\lambda}_1 < \beta_2 = \lambda_1 \mu \bar{\lambda}_1 \mu.$$

Hence,  $\lambda_1 \mu | \beta_1 = \bar{\mu} \lambda_1 \mu \bar{\lambda}_1$  and therefore  $\lambda_1 \mu$  is a left factor of  $\beta_1$  of the same length as  $\bar{\mu} \lambda_1$ . Thus we get the word equation;

$$\lambda_1 \mu = \bar{\mu} \lambda_1. \tag{4}$$

Therefore, case (3) of the theorem is satisfied.

Suppose secondly that  $\beta_2 = \lambda_1 \mu \overline{\lambda_1} \mu < \beta_1 = \overline{\mu} \lambda_1 \mu \overline{\lambda_1}$ . Since  $\beta_2$  is the greatest word in  $\langle\langle \beta_2 \rangle\rangle$  and  $\overline{\mu} \lambda_1 \mu \overline{\lambda_1} \in \langle\langle \beta_2 \rangle\rangle$  we have:

$$\overline{\mu} \lambda_1 \mu \overline{\lambda_1} < \beta_2 = \lambda_1 \mu \overline{\lambda_1} \mu < \beta_1 = \overline{\mu} \lambda_1 \mu \overline{\lambda_1}.$$

Hence,  $\overline{\mu} \lambda_1 | \beta_2 = \lambda_1 \mu \overline{\lambda_1} \mu$  and once again we get the word equation (4).

(c) If  $|\mu| > |\delta|$  then  $|\lambda| < |\delta|$  and hence  $\overline{\mu} = \delta \mu_1$  and  $\overline{\delta} = \delta_1 \lambda$  from which one can easily conclude that  $\mu_1 = \delta_1$ ,  $\beta_1 = \overline{\mu_1} \overline{\lambda} \mu_1 \lambda$  and  $\beta_2 = \lambda \mu_1 \lambda \overline{\mu_1}$ . Suppose firstly that  $\beta_2 = \lambda \mu_1 \lambda \overline{\mu_1} < \beta_1 = \overline{\mu_1} \overline{\lambda} \mu_1 \lambda$ . Since  $\beta_2$  is the greatest word in  $\langle\langle \beta_2 \rangle\rangle$  and  $\overline{\mu_1} \overline{\lambda} \mu_1 \lambda \in \langle\langle \beta_2 \rangle\rangle$  we have:

$$\overline{\mu_1} \overline{\lambda} \mu_1 \lambda < \beta_2 = \lambda \mu_1 \lambda \overline{\mu_1} < \beta_1 = \overline{\mu_1} \overline{\lambda} \mu_1 \lambda.$$

Hence,  $\overline{\mu_1} \overline{\lambda} | \beta_2 = \lambda \mu_1 \lambda \overline{\mu_1}$  and we get the word equation;

$$\lambda \mu_1 = \overline{\mu_1} \overline{\lambda}. \quad (5)$$

Therefore, case (4) of the theorem is satisfied.

On the other hand, if  $\beta_1 = \overline{\mu_1} \overline{\lambda} \mu_1 \lambda < \beta_2 = \lambda \mu_1 \lambda \overline{\mu_1}$  then

$$\lambda \overline{\mu_1} \overline{\lambda} \mu_1 < \beta_1 = \overline{\mu_1} \overline{\lambda} \mu_1 \lambda < \beta_2 = \lambda \mu_1 \lambda \overline{\mu_1}.$$

Hence,  $\lambda | \beta_1 = \overline{\mu_1} \overline{\lambda} \mu_1 \lambda$ . Thus  $\lambda$  is an initial subword of  $\overline{\mu_1} \overline{\lambda}$ . Therefore, we get the word equation

$$\overline{\mu_1} \overline{\lambda} = \lambda \eta, \quad (6)$$

which is of the same type as equation (3) and case (5) of the theorem is satisfied.

Subcase 2:  $\beta_1 = \overline{\mu} \lambda \in \mathbf{L}_n(\overline{C})$  and  $\beta_2 = \lambda \mu = \delta \overline{\delta}$  for odd  $\delta$ .

(a) If  $|\mu| = |\delta|$  then  $\lambda = \delta$  and  $\mu = \overline{\delta}$  and hence,  $\lambda = \overline{\mu}$ . But then  $\beta_1 = \lambda^2$  which is a contradiction to the fact that  $\beta_1$  is primitive.

(b) If  $|\mu| < |\delta|$  then  $|\lambda| > |\delta|$ . Hence,  $\lambda = \delta \lambda_1$  and  $\overline{\delta} = \delta_1 \mu$  where  $\lambda_1 = \delta_1$ . Therefore,  $\beta_1 = \overline{\mu} \lambda_1 \mu \lambda_1$  and  $\beta_2 = \overline{\lambda_1} \overline{\mu} \lambda_1 \mu$ . Now, if  $\beta_1 < \beta_2$  then;

$$\overline{\lambda_1} \overline{\mu} \lambda_1 \mu < \beta_1 = \overline{\mu} \lambda_1 \mu \lambda_1 < \beta_2 = \overline{\lambda_1} \overline{\mu} \lambda_1 \mu.$$

Hence, we get the word equation

$$\overline{\mu} \lambda_1 = \overline{\lambda_1} \overline{\mu}, \quad (7)$$

which is of the same type as equation (3) and case (6) of the theorem is satisfied.

Similarly, if  $\beta_2 < \beta_1$  then

$$\overline{\mu} \lambda_1 \mu \overline{\lambda_1} < \beta_2 = \overline{\lambda_1} \overline{\mu} \lambda_1 \mu < \beta_1 = \overline{\mu} \lambda_1 \mu \lambda_1.$$

Therefore we get once again the same word equation 7.

(c) If  $|\mu| > |\delta|$  then  $|\lambda| < |\delta|$ . Letting  $\delta = \lambda \delta_1$  and  $\mu = \mu_1 \overline{\delta}$ , one can easily see that  $\delta_1 = \mu_1$  and therefore  $\beta_1 = \overline{\mu_1} \lambda \mu_1 \lambda$  and  $\beta_2 = \lambda \mu_1 \overline{\lambda} \overline{\mu_1}$ . If  $\beta_1 < \beta_2$  then;

$$\lambda \mu_1 \lambda \overline{\mu_1} < \beta_1 = \overline{\mu_1} \lambda \mu_1 \lambda < \beta_2 = \lambda \mu_1 \overline{\lambda} \overline{\mu_1}$$

and hence  $\lambda\mu_1 = \bar{\mu}_1\lambda$  which is a word equation of the same type as 4 and case (7) of the theorem is satisfied.

If  $\beta_2 < \beta_1$  then;

$$\bar{\mu}_1\lambda\mu_1\bar{\lambda} < \beta_2 = \lambda\mu_1\bar{\lambda}\bar{\mu}_1 < \beta_1 = \bar{\mu}_1\lambda\mu_1\lambda$$

once again we end with the same word equation  $\lambda\mu_1 = \bar{\mu}_1\lambda$ .

III.  $\beta_1 \notin \mathbf{L}_n(\bar{C})$  and  $\beta_2 \notin \mathbf{L}_n(\bar{C})$ .

By Theorem 2.8 [Lu],  $\beta_1 = \bar{\mu}\lambda = \delta\bar{\delta}$  and  $\beta_2 = \lambda\mu = \rho\bar{\rho}$  for odd  $\delta$  and  $\rho$ . We shall prove that this case can not happen. Clearly we have  $|\delta| = |\rho|$ . Consider the following subcases:

Subcase 1:  $|\lambda| = |\delta| = |\rho|$ .

We have  $\bar{\mu} = \delta, \lambda = \bar{\delta}$  and  $\lambda = \rho, \mu = \bar{\rho}$ . Thus  $\bar{\mu} = \delta = \bar{\lambda} = \bar{\rho} = \mu$ . Therefore  $\lambda$  and  $\mu$  are powers of  $M$  which implies that  $\beta_1$  and  $\beta_2$  are not primitive, a contradiction.

Subcase 2:  $|\lambda| < |\delta| = |\rho|$ . Hence,  $|\mu| > |\delta| = |\rho|$ .

We have  $\bar{\mu} = \delta\mu_2, \bar{\delta} = \delta_1\lambda$  and  $\rho = \lambda\rho_1, \mu = \mu_1\bar{\rho}$ . It is easily seen that  $\mu_2 = \delta_1$  and  $\rho_1 = \mu_1$ . Hence, we may write  $\bar{\mu} = \delta\mu_2, \bar{\delta} = \mu_2\lambda$  and  $\rho = \lambda\mu_1, \mu = \mu_1\bar{\rho}$ . In particular,  $\bar{\mu} = \mu_2\lambda\mu_2$ , and  $\mu = \mu_1\bar{\lambda}\mu_1$ . Combining both these two equations we see that  $\mu = \mu_2\lambda\mu_2 = \mu_1\bar{\lambda}\mu_1$ . This implies that  $\mu_1 = \mu_2$  and  $\lambda = \bar{\lambda} = M^l, l = |\lambda|$ . But this contradicts the fact that the first letter of  $\beta_2 = \lambda\mu$ , which is primitive and the greatest word in  $\langle\langle\beta_2\rangle\rangle$ , must be  $R$  and not  $M$ .

Subcase 3:  $|\lambda| > |\delta| = |\rho|$ . Hence,  $|\mu| < |\delta| = |\rho|$ .

Then  $\delta = \bar{\mu}\delta_1, \lambda = \lambda_2\bar{\delta}$  and  $\lambda = \rho\lambda_1, \bar{\rho} = \rho_1\mu$ . Once again it is easily seen that  $\delta_1 = \lambda_2$  and  $\lambda_1 = \rho_1$ . Hence, we conclude that  $\lambda = \lambda_2\mu\bar{\lambda}_2 = \bar{\lambda}_1\mu\lambda_1$ . Therefore  $\mu = \mu = M^l, l = |\mu|$  and this contradicts the fact that the first letter of  $\beta_1 = \bar{\mu}\lambda$ , which is primitive and the greatest word in  $\langle\langle\beta_1\rangle\rangle$ , must be  $R$  and not  $M$ . ■

**Lemma 4** If  $\beta = LAL$ , where  $L, A \in \mathbf{W}$  then  $\beta \notin \mathbf{L}_n(\bar{C})$  and  $\beta \notin \mathbf{L}_n(C)$ .

**Proof.** Both cases are similar so we prove the first one, namely;  $\beta \notin \mathbf{L}_n(\bar{C})$ . Suppose the contrary, then  $\beta = LAL \in \mathbf{L}_n(\bar{C})$  implies that  $\beta = LAL > L$ . Hence, in  $\bar{C}$ -order,  $L$  must be an odd word. On the other hand,  $\bar{\beta} = \bar{L}\bar{A}\bar{L} < \bar{L}$  implies that, in  $\bar{C}$ -order,  $\bar{L}$  is even. Since  $L$  and  $\bar{L}$  have the same parity we get a contradiction. ■

Now, the  $\bar{C}$ -lexicity of  $\beta_1$  or  $\beta_2$  which was emphasized in Theorem 1 allows us to make the following refinement of the previous theorem.

**Theorem 2** If  $\langle\beta_1\bar{\beta}_1\rangle = \langle\beta_2\bar{\beta}_2\rangle$  where  $\beta_1$  and  $\beta_2$  are the greatest words in  $\langle\langle\beta_1\rangle\rangle \in \mathbf{M}(n)$  and  $\langle\langle\beta_2\rangle\rangle \in \mathbf{M}(n)$ , respectively, and  $\langle\langle\beta_1\rangle\rangle \neq \langle\langle\beta_2\rangle\rangle$  then one of the following occurs;

- (a)  $\beta_1 = \bar{\mu}\lambda = \bar{\mu}^2X_1 \in \mathbf{L}_n(\bar{C}), \beta_2 = \lambda\mu = \bar{\mu}X_1\mu \in \mathbf{L}_n(\bar{C})$ .
- (b)  $\beta_1 = \bar{\mu}_1\bar{\lambda}\mu_1\lambda = \lambda X_1\bar{\lambda}\bar{\lambda}X_1\lambda$  and  $\beta_2 = \lambda\mu_1\bar{\lambda}\bar{\mu}_1 = \lambda\bar{\lambda}X_1\lambda\bar{\lambda}X_1 \in \mathbf{L}_n(\bar{C})$ .

**Proof.** By Theorem 1, we ought to consider the following cases:

I.  $\beta_2 = \lambda\mu \in \mathbf{L}_n(\overline{C})$ ,  $\beta_1 = \overline{\mu}\lambda \in \mathbf{L}_n(\overline{C})$  and  $\lambda\mu = \overline{\mu}\eta$ . Applying the conclusion of Proposition 3 by substituting  $X, Y$  and  $Z$  as shown in the following table (namely:  $X = \lambda, Y = \mu$  and  $Z = \eta$ ):

$$\begin{array}{ccc} X & Y & Z \\ \lambda & \mu & \eta \end{array}$$

we have the following subcases:

Subcase 1: If  $|\lambda| = |\mu| = |\eta|$  then  $\lambda = \overline{\eta}$  and we get the word equation:  $\lambda\mu = \overline{\mu}\lambda$ . Applying Proposition 1 by substituting,

$$\begin{array}{cc} Z & W \\ \lambda & \mu \end{array}$$

it follows that  $\lambda = \overline{\mu}$  and therefore  $\beta_1 = \overline{\mu}\lambda = \lambda^2$  is not primitive, a contradiction.

Subcase 2: If  $|\lambda| = |\eta| < |\mu|$  then either (a)  $\overline{\lambda} = Y_0X_0$ ,  $\eta = \overline{X_0Y_0}$  and  $\mu = (Y_0X_0\overline{Y_0X_0})^{n_1}Y_0$ ,  $n_1 \geq 1$ , which implies:

$$\begin{aligned} \beta_2 &= \lambda\mu = \overline{Y_0X_0}(Y_0X_0\overline{Y_0X_0})^{n_1}Y_0 = (\overline{Y_0X_0}Y_0X_0)^{n_1}\overline{Y_0X_0}Y_0, \\ \beta_1 &= \overline{\mu}\lambda = (\overline{Y_0X_0}Y_0X_0)^{n_1}\overline{Y_0Y_0X_0}. \end{aligned}$$

By the previous lemma this leads to a contradiction, since  $\beta_1 \in \mathbf{L}_n(\overline{C})$  have the same left and right factor  $\overline{Y_0X_0}$ , or (b)  $\lambda = Y_0X_0$ ,  $\eta = \overline{X_0Y_0}$  and  $\mu = (\overline{Y_0X_0}Y_0X_0)^{n_2}\overline{Y_0X_0}Y_0$  which implies:

$$\begin{aligned} \beta_2 &= \lambda\mu = Y_0X_0(\overline{Y_0X_0}Y_0X_0)^{n_2}\overline{Y_0X_0}Y_0 = (Y_0X_0\overline{Y_0X_0})^{n_2+1}Y_0, \\ \beta_1 &= \overline{\mu}\lambda = (Y_0X_0\overline{Y_0X_0})^{n_2}Y_0X_0\overline{Y_0Y_0X_0} = (Y_0X_0\overline{Y_0X_0})^{n_2+1}X_0. \end{aligned}$$

But then we have a contradiction, according the previous lemma, since  $\beta_2 \in \mathbf{L}_n(\overline{C})$ .

Subcase 3: If  $|\lambda| = |\eta| > |\mu|$  then  $\lambda = \overline{\mu}X_1$  and  $\eta = X_1\mu$  which implies

$$\begin{aligned} \beta_2 &= \overline{\mu}X_1\mu, \\ \beta_1 &= \overline{\mu\mu}X_1 = \overline{\mu}^2X_1. \end{aligned}$$

This case is possible.

II.  $\beta_1 = \overline{\mu}\lambda_1\mu\overline{\lambda_1}$ ,  $\beta_2 = \lambda_1\mu\overline{\lambda_1}\mu$  and  $\lambda_1\mu = \overline{\mu}\lambda_1$ . Applying Proposition 2 by substituting

$$\begin{array}{cc} X & Y \\ \lambda_1 & \mu \end{array}$$

we conclude that  $\beta_1 = \overline{\mu}\lambda_1\mu\overline{\lambda_1}$  is not primitive, a contradiction.

III.  $\beta_2 = \lambda\mu_1\lambda\overline{\mu_1} \in \mathbf{L}_n(\overline{C})$ ,  $\beta_1 = \overline{\mu_1}\lambda\mu_1\lambda$  and  $\lambda\mu_1 = \overline{\mu_1}\lambda$ . If  $|\lambda| \neq |\mu_1|$  then applying Proposition 1 by the substitution

$$\begin{array}{cc} Z & W \\ \lambda & \mu_1 \end{array}$$

we get that  $\lambda$  and  $\mu_1$  are powers of  $M$  and therefore  $\beta_1$  and  $\beta_2$  are not primitive which is a contradiction, or  $\lambda = (\overline{L}L)^{n_1}\overline{L}$ ,  $\mu_1 = (L\overline{L})^{n_2}L$  and hence,  $\beta_1 = \overline{\mu_1}\overline{\lambda}\mu_1\lambda = (\overline{L}L)^{n_1+n_2+1}(L\overline{L})^{n_1+n_2+1}$  and  $\beta_2 = \lambda\mu_1\lambda\overline{\mu_1} = (\overline{L}L)^{2n_1+n_2+1}\overline{L}^2(L\overline{L})^{n_2}$ .

If  $|\lambda| = |\mu_1|$  then  $\lambda = \overline{\mu_1}$  and hence;

$$\begin{aligned}\beta_2 &= \lambda\mu_1\lambda\overline{\mu_1} = \overline{\mu_1}\mu_1\overline{\mu_1}\mu_1, \\ \beta_1 &= \overline{\mu_1}\overline{\lambda}\mu_1\lambda = \overline{\mu_1}\mu_1\mu_1\overline{\mu_1}.\end{aligned}$$

In both cases, by the previous lemma this leads to a contradiction since  $\beta_2 \in \mathbf{L}_n(\overline{C})$ .

IV.  $\beta_1 = \overline{\mu_1}\overline{\lambda}\mu_1\lambda$ ,  $\beta_2 = \lambda\mu_1\lambda\overline{\mu_1} \in \mathbf{L}_n(\overline{C})$  and  $\overline{\mu_1}\overline{\lambda} = \lambda\eta$ . Applying the conclusion of Proposition 3 making the following substitution

$$\begin{array}{ccc} X & Y & Z \\ \overline{\mu_1} & \overline{\lambda} & \eta \end{array}$$

we have the following subcases:

Subcase 1: If  $|\mu_1| = |\lambda| = |\eta|$  then  $\overline{\mu_1} = \overline{\eta}$  and we get the word equation  $\lambda\mu_1 = \overline{\mu_1}\overline{\lambda}$ . Applying Proposition 1 by substituting

$$\begin{array}{cc} Z & W \\ \lambda & \mu_1 \end{array}$$

it follows that  $\lambda = \overline{\mu_1}$  and therefore;  $\beta_1 = \overline{\mu_1}\mu_1\mu_1\overline{\mu_1}$  and  $\beta_2 = \overline{\mu_1}\mu_1\overline{\mu_1}\mu_1$  which is similar to a previous case and can not occur.

Subcase 2: If  $|\mu_1| = |\eta| < |\lambda|$  then either  $\mu_1 = Y_0X_0$ ,  $\eta = \overline{X_0}Y_0$  and  $\overline{\lambda} = (Y_0X_0\overline{Y_0X_0})^{n_1}Y_0$ ,  $n_1 \geq 1$ , which implies:

$$\begin{aligned}\beta_1 &= \overline{\mu_1}\overline{\lambda}\mu_1\lambda = (\overline{Y_0X_0}Y_0X_0)^{n_1}\overline{Y_0X_0}Y_0(Y_0X_0\overline{Y_0X_0})^{n_1}Y_0X_0\overline{Y_0}, \\ \beta_2 &= \lambda\mu_1\lambda\overline{\mu_1} = (\overline{Y_0X_0}Y_0X_0)^{n_1}\overline{Y_0}Y_0X_0(\overline{Y_0X_0}Y_0X_0)^{n_1}\overline{Y_0}Y_0X_0.\end{aligned}$$

or  $\overline{\mu_1} = Y_0X_0$ ,  $\eta = \overline{X_0}Y_0$  and  $\overline{\lambda} = (\overline{Y_0X_0}Y_0X_0)^{n_2}\overline{Y_0X_0}Y_0$  which implies:

$$\begin{aligned}\beta_1 &= \overline{\mu_1}\overline{\lambda}\mu_1\lambda = (Y_0X_0\overline{Y_0X_0})^{n_2+1}Y_0(\overline{Y_0X_0}Y_0X_0)^{n_2+1}\overline{Y_0}, \\ \beta_2 &= \lambda\mu_1\lambda\overline{\mu_1} = (Y_0X_0\overline{Y_0X_0})^{n_2}Y_0X_0\overline{Y_0}(\overline{Y_0X_0}Y_0X_0)^{n_2+1}\overline{Y_0}Y_0X_0.\end{aligned}$$

Both cases can not happen, by the previous lemma, since  $\beta_2 = \lambda\mu_1\lambda\overline{\mu_1} \in \mathbf{L}_n(\overline{C})$ .

Subcase 3: If  $|\mu_1| = |\eta| > |\lambda|$  then  $\overline{\mu_1} = \lambda X_1$  and  $\eta = X_1\overline{\lambda}$  which implies:

$$\begin{aligned}\beta_1 &= \overline{\mu_1}\overline{\lambda}\mu_1\lambda = \lambda X_1\overline{\lambda\lambda X_1}\lambda, \\ \beta_2 &= \lambda\mu_1\lambda\overline{\mu_1} = \lambda\overline{\lambda X_1}\lambda\lambda X_1.\end{aligned}$$

This case is possible.

V.  $\beta_1 = \overline{\mu}\lambda_1\mu\lambda_1$ ,  $\beta_2 = \overline{\lambda_1}\overline{\mu}\lambda_1\mu$  and  $\overline{\mu}\lambda_1 = \overline{\lambda_1}\overline{\mu}$ . Making the substitution

$$\begin{array}{cc} X & Y \\ \overline{\mu} & \lambda_1 \end{array}$$

in Proposition 2, it follows that  $\beta_2 = \overline{\lambda_1}\overline{\mu}\lambda_1\mu$  is not primitive, a contradiction.

VI.  $\beta_1 = \overline{\mu_1}\lambda\mu_1\lambda$ ,  $\beta_2 = \lambda\mu_1\overline{\lambda}\overline{\mu_1}$  and  $\lambda\mu_1 = \overline{\mu_1}\lambda$ . By Proposition 2, we get that  $\beta_2 = \lambda\mu_1\overline{\lambda}\overline{\mu_1}$  is not primitive, a contradiction. ■

## 5 Concluding remarks

We conclude by some examples illustrating the last theorem. For  $n = 4$  and by the computations which was carried out in [10, p. 2189], we have  $|\mathbf{M}(4)| = 10$ . The elements  $\langle\langle\beta\rangle\rangle$  of  $\mathbf{M}(4)$  for which  $\beta$  is the greatest word in  $\langle\langle\beta\rangle\rangle$  are

$$\begin{aligned} &\langle\langle R^3M \rangle\rangle, \langle\langle R^3L \rangle\rangle, \langle\langle R^2M^2 \rangle\rangle, \langle\langle R^2ML \rangle\rangle, \langle\langle R^2LM \rangle\rangle, \\ &\langle\langle R^2L^2 \rangle\rangle, \langle\langle RM^3 \rangle\rangle, \langle\langle RM^2L \rangle\rangle, \langle\langle RMRL \rangle\rangle, \langle\langle RMLM \rangle\rangle. \end{aligned}$$

Now, if  $\langle\langle\beta\rangle\rangle \in \{\langle\langle R^3M \rangle\rangle, \langle\langle R^2ML \rangle\rangle\}$  then  $\langle\beta\bar{\beta}\rangle = \langle R^3ML^3M \rangle$ , if  $\langle\langle\beta\rangle\rangle \in \{\langle\langle R^3L \rangle\rangle, \langle\langle R^2L^2 \rangle\rangle\}$  then  $\langle\beta\bar{\beta}\rangle = \langle R^4L^4 \rangle$ , and if  $\langle\langle\beta\rangle\rangle \in \{\langle\langle R^2M^2 \rangle\rangle, \langle\langle RM^2L \rangle\rangle\}$  then  $\langle\beta\bar{\beta}\rangle = \langle R^2M^2L^2M^2 \rangle$ . Notice that all the three pairs satisfy condition (a) of Theorem 2.

On the other hand, if  $\lambda = R^2M$  and  $X_1 = LM$  then  $\beta_1 = R^2MLML^2ML^2MRMR^2M \in \mathbf{M}(14)$  and  $\beta_2 = R^2ML^2MRMR^2MR^2MLM \in \mathbf{M}(14)$  satisfy condition (b) of Theorem 2. Notice that  $\beta_1$  and  $\beta_2$  are greatest words in  $\langle\langle\beta_1\rangle\rangle$  and  $\langle\langle\beta_2\rangle\rangle$ , respectively, and  $\beta_2 \in \mathbf{L}_n(\overline{C})$ .

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## References

- [1] Abboud, E.: Algorithms for producing and ordering lexical and nonlexical sequences out of one element, *Ann. Comb.*, 15(1), 1–17 (2011)
- [2] Brucks, K. M.: MSS sequences, coloring of necklaces, and periodic points of  $f(z) = z^2 - 2$ . *Adv. Appl. Math.*, 8, 434–445 (1987).
- [3] Chen, W. Y. C., Louck, J. D.: Necklaces, MSS sequences and DNA sequences. *Adv. Appl. Math.*, 18, 18–32 (1997).
- [4] Chen, W. Y. C., Louck, J. D., Wang, J.: Adjacency and parity relations of words in discrete dynamical systems. *J. Combin. Theory Ser. A*, 91, 476–508 (2000).
- [5] Chen, W. Y. C., Wang, J.: Decomposition of necklaces. *Ann. Comb.*, 5(3–4), 271–283 (2001).
- [6] Dai, W., Lu, K., Wang, J.: Combinatorics on words in symbolic dynamics: the quadratic map. *Acta Math. Sin., Engl. Series*, 24(12), 1985–1994 (2008).
- [7] Dai, W., Lu, K., Wang, J.: Combinatorics on words in symbolic dynamics: the antisymmetric cubic map. *Acta Math. Sin., Engl. Series*, 24(11), 1817–1834 (2008).
- [8] G. Helmberg and L. Sun, Maximal words connected with unimodal maps, *Order* 4 (1988) 351–380.

- [9] Lothaire, M.: Combinatorics on Words, Encyclopedia of Mathematics and Its Applications 17, G.-C. Rota, Ed., Addison–Wesley, Reading, 1983.
- [10] K. B. Lu, Some combinatorial properties of words in discrete dynamical systems from antisymmetric cubic maps, Acta Mathematica Sinica, English Series, 29(11), pp 2181-2192 (2013).